Some conceptual issues involving probability in quantum mechanics*

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1 Introduction

The issue of the completeness of quantum mechanics has been a subject of intense research for almost a century. One of the most influential papers is undoubtedly that of Eintein, Podolski and Rosen [Einstein et al. 1935], where after analyzing entangled two-particle states they concluded that quantum mechanics could not be considered a complete theory. In 1964 John Bell showed that not only was quantum mechanics incomplete but, if one wanted a complete description of reality that was local, one would obtain correlations that are incompatible with the ones predicted by quantum mechanics [Bell 1987]. This happens because some quantum mechanical states do not allow for the existence of joint probability distributions of all the possible outcomes of experiments. If a joint distribution exists, then one could consistently create a local hidden variable that would factor this distribution. The nonexistence of local hidden variables that would "complete" quantum mechanics, hence the nonexistence of joint probability distributions, was verified experimentally in 1982 by Aspect, Dalibard and Roger [Aspect at al. 1982], when they showed, in a series of beautifully designed experiments, that an entangled photon state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle), \tag{1}$$

(where $|+-\rangle \equiv |+\rangle_A \otimes |-\rangle_B$ represents, for example, two photons A and B with helicity $+1$ and -1, respectively) violates the Clauser-Horne-Shimony-Holt form

*It is a pleasure to dedicate this article to Arthur Fine. The subject of our paper is close to one of Arthur's best known articles on the foundations of physics [Fine 1982].

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of Bell's inequalities [Clauser et al. 1969], as predicted by quantum mechanical computations. More recently, Weihs et al. confirmed Aspect's experiment with a truly random selection of the polarization angles, thus with a more strict nonlocality criteria satisfied [Weihs et al. 1998]. We note that the proof that the Clauser et al. form of Bell's inequalities implies the existence of a joint probability distribution of the observable random variables is the mains result in [Fine 1982].

The nonexistence of joint probability distributions also comes into play in the consistent-history interpretation of quantum mechanics. In this interpretation, each sequence of properties for a given quantum mechanical system represents a possible history for this system, and a set of such histories is called a family of histories [Gell-Mann and Hartle 1990]. A family of *consistent* histories is one that has a joint probability distribution for all possible histories in this family, with the joint probability distribution defined as any probability measure on the space of all histories. One can easily show that quantum mechanics implies the nonexistence of such probability functions for some families of histories. Families of histories that do not have a joint probability distribution are called inconsistent histories.

Another important example, also related to the nonexistence of a joint probability distribution, is the famous Kochen-Specker theorem, that shows that a given hidden-variable theory that is consistent with the quantum mechanical results has to be contextual [Kochen and Specker 1967], i.e., the hidden variable has to depend on the values of the actual experimental settings, regardless of how far apart the actual components of the experiment are located (throughout this paper, we will use interchangeably the concepts of local and noncontextual hidden variables; for a detailed discussion, see [Suppes and Zanotti 1976] and [D'Espagnat 1989]).

More recently, a marriage between Bell's inequalities and the Kochen-Specker theorem led to the Greenberger-Horne-Zeilinger (GHZ) theorem. The GHZ theorem shows that if one assumes that one can consistently assign values to the outcomes of a measurement before the measure is performed, a mathematical contradiction arises [Greenberger et al. 1989] — once again, having a complete data table would allow us to compute the joint probability distribution, so we conclude that no joint distribution exists that is consistent with quantum mechanical results. In this paper, we propose the usage of nonmonotonic upper probabilities as a tool to derive consistent joint upper probabilities for the contextual hidden variables.

2 The GHZ Theorem

In 1989 Greenberger, Horne and Zeilinger (GHZ) proved that if the quantum mechanical predictions for entangled states are correct, then the assumption that there exist noncontextual hidden variables that can accommodate those predictions leads to contradictions [Greenberger et al. 1989]. Their proof of the incompatibility of noncontextual hidden variables with quantum mechanics is

now known as the GHZ theorem. This theorem proposes a new test for quantum mechanics based on correlations between more than two particles. What makes the GHZ theorem distinct from Bell's inequalities is the fact that they use only perfect correlations. The argument for the GHZ theorem, as stated by Mermin [Mermin 1990a], goes as follows. We start with a three-particle entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|+\rangle_2|-\rangle_3 + |-\rangle_1|-\rangle_2|+\rangle_3), \tag{2}$$

where we use a notation similar to that of equation (1). This state is an eigenstate of the following spin operators:

$$\hat{\mathbf{A}} = \hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}, \quad \hat{\mathbf{B}} = \hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}, \tag{3}$$

$$\hat{\mathbf{C}} = \hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}, \quad \hat{\mathbf{D}} = \hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}. \tag{4}$$

If we compute the expected values for the correlations above, we obtain at once that $E(\hat{\mathbf{A}}) = E(\hat{\mathbf{B}}) = E(\hat{\mathbf{C}}) = 1$ and $E(\hat{\mathbf{D}}) = -1$. Let us now suppose that the value of the spin for each particle is dictated by a hidden variable λ, and let us call this value $s_{ij}(\lambda)$, where $i = 1...3$ and $j = x, y$. Then, we have that

$$E(\hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}) = (s_{1x}s_{2y}s_{3y})(s_{1y}s_{2x}s_{3y})(s_{1y}s_{2y}s_{3x}) \tag{5}$$

$$= s_{1x}s_{2x}s_{3x}(s_{1y}^2 s_{2y}^2 s_{3y}^2). \tag{6}$$

Since the $s_{ij}(\lambda)$ can only be 1 or -1, we obtain

$$E(\hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}) = s_{1x}s_{2x}s_{3x} = E(\hat{\mathbf{D}}). \tag{7}$$

But (5) implies that $E(\hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}) = 1$ whereas (7) implies $E(\hat{\mathbf{A}}\hat{\mathbf{B}}\hat{\mathbf{C}}) = E(\hat{\mathbf{D}}) = -1$, a clear contradiction. It is clear from the above derivation that one could avoid contradictions if we allowed the value of λ to depend on the experimental setup, i.e., if we allowed λ to be a contextual hidden variable. In other words, what the GHZ theorem proves is that noncontextual hidden variables cannot reproduce quantum mechanical predictions.

This striking characteristic of GHZ's predictions, however, has a major problem. How can one verify experimentally predictions based on correlation-one statements, since experimentally one cannot obtain events perfectly correlated? This problem was also present on Bell's original paper, where he considered cases where the correlations were one. To "avoid Bell's experimentally unrealistic restrictions", Clauser, Horne, Shimony and Holt [Clauser et al. 1969] derived a new set of inequalities that would take into account imperfections in the measurement process. However, Bell's inequalities are quite different from the GHZ case, where it is necessary to have experimentally unrealistic perfect correlations. This can be seen from the following theorem (a version for a 4 particle entangled system is found in [Suppes et al. 1998]).

Theorem 1 Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be three ± 1 random variables and let

(i) $E(\mathbf{A}) = E(\mathbf{B}) = E(\mathbf{C}) = 1$,

(ii) $E(\mathbf{ABC}) = -1$,

then (i) and (ii) imply a contradiction.

Proof: By definition

$$E(\mathbf{A}) = P(a) - P(\overline{a}), \tag{8}$$

where we use a notation where a is $\mathbf{A} = 1$, $\overline{a}$ is $\mathbf{A} = -1$, and so on. Since $0 \leq P(a), P(\overline{a}) \leq 1$, it follows at once from (i) that

$$P(a) = 1 \tag{9}$$

and similarly

$$P(b) = P(c) = 1. \tag{10}$$

Using again the definition of expectation and the inequalities $P(\overline{a}bc) \leq P(\overline{a}) = 0$, etc., we have

$$\begin{aligned} E(\mathbf{ABC}) &= P(abc) + P(\overline{ab}c) + P(a\overline{bc}) + P(\overline{a}b\overline{c}) \\ &= P(abc) - [P(\overline{a}bc) + P(a\overline{b}c) + P(ab\overline{c}) + P(\overline{a}\overline{b}\overline{c})] \\ &= 1, \end{aligned} \tag{11}$$

from (9) and (10), since all but the first term on the right is 0, and thus by conservation of probability $P(ABC) = 1$. But (11) contradicts (ii).

It is important to note that if we could measure all the random variables simultaneously, we would have a joint distribution. The existence of a joint probability distribution is a necessary and sufficient condition for the existence of a noncontextual hidden variable [Suppes and Zanotti 1981]. Hence, if the quantum mechanical GHZ correlations are obtained, then no noncontextual hidden variable exists. However, this abstract version of the GHZ theorem still involves probability-one statements. On the other hand, the correlations present in the GHZ state are so strong that even if we allow for experimental errors, the non-existence of a joint distribution can still be verified, as we show in the following theorem [Barros and Suppes 2000].

Theorem 2 If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are three ± 1 random variables, a joint probability distribution exists for the given expectations $E(\mathbf{A})$, $E(\mathbf{B})$, $E(\mathbf{C})$, and $E(\mathbf{ABC})$ if and only if the following inequalities are satisfied:

$$-2 \leq E(\mathbf{A}) + E(\mathbf{B}) + E(\mathbf{C}) - E(\mathbf{ABC}) \leq 2, \tag{12}$$

$$-2 \leq -E(\mathbf{A}) + E(\mathbf{B}) + E(\mathbf{C}) + E(\mathbf{ABC}) \leq 2, \tag{13}$$

$$-2 \leq E(\mathbf{A}) - E(\mathbf{B}) + E(\mathbf{C}) + E(\mathbf{ABC}) \leq 2, \tag{14}$$

$$-2 \leq E(\mathbf{A}) + E(\mathbf{B}) - E(\mathbf{C}) + E(\mathbf{ABC}) \leq 2. \tag{15}$$

Proof: First we prove necessity. Let us assume that there is a joint probability distribution consisting of the eight atoms abc, $ab\overline{c}$, $a\overline{b}c$, $...\overline{a}\overline{b}\overline{c}$. Then,

$$E(\mathbf{A}) = P(a) - P(\overline{a}),$$

where

$$P(a) = P(abc) + P(a\overline{b}c) + P(ab\overline{c}) + P(a\overline{b}\overline{c}),$$

and

$$P(\overline{a}) = P(\overline{a}bc) + P(\overline{a}\overline{b}c) + P(\overline{a}b\overline{c}) + P(\overline{a}\overline{b}\overline{c}).$$

Similar equations hold for $E(\mathbf{B})$ and $E(\mathbf{C})$. For $E(\mathbf{ABC})$ we obtain

$$\begin{aligned} E(\mathbf{ABC}) &= P(\mathbf{ABC} = 1) - P(\mathbf{ABC} = -1) \\ &= P(abc) + P(a\overline{b}\overline{c}) + +P(\overline{a}\overline{b}c) + P(\overline{a}b\overline{c}) \\ &\quad -[P(a\overline{b}c) + P(ab\overline{c}) + P(\overline{a}bc) + P(\overline{a}\overline{b}\overline{c})]. \end{aligned}$$

Corresponding to the first inequality above, we now sum over the probability expressions for the expectations

$$F = E(\mathbf{A}) + E(\mathbf{B}) + E(\mathbf{C}) - E(\mathbf{ABC}),$$

and obtain the expression

$$\begin{aligned} F &= 2[P(abc) + P(\overline{a}bc) + P(a\overline{b}c) + P(ab\overline{c})] \\ &\quad -2[P(\overline{a}\overline{b}\overline{c}) + P(\overline{a}\overline{b}c) + P(\overline{a}b\overline{c}) + P(a\overline{b}\overline{c})], \end{aligned}$$

and since all the probabilities are nonnegative and sum to ≤ 1, we infer at once inequality (12). The derivation of the other three inequalities is very similar.

To prove the converse, i.e., that these inequalities imply the existence of a joint probability distribution, is slightly more complicated. We restrict ourselves to the symmetric case

$$P(a) = P(b) = P(c) = p,$$

$$P(\mathbf{ABC} = 1) = q$$

and thus

$$E(\mathbf{A}) = E(\mathbf{B}) = E(\mathbf{C}) = 2p - 1,$$

$$E(\mathbf{ABC}) = 2q - 1.$$

In this case, (12) can be written as

$$0 \leq 3p - q \leq 2,$$

while the other three inequalities yield just $0 \leq p + q \leq 2$. Let

$$x = P(\overline{a}bc) = P(a\overline{b}c) = P(ab\overline{c}),$$

$$y = P(\overline{a}\overline{b}c) = P(\overline{a}b\overline{c}) = P(a\overline{b}\overline{c}),$$

$$z = P(abc),$$

and

$$w = P(\overline{a}\overline{b}\overline{c}).$$

It is easy to show that on the boundary $3p = q$ defined by the inequalities the values $x = 0$, $y = q/3$, $z = 0$, $w = 1 - q$ define a possible joint probability distribution, since $3x + 3y + z + w = 1$. On the other boundary, $3p = q + 2$ a possible joint distribution is $x = (1 - q)/3$, $y = 0$, $z = q$, $w = 0$. Then, for any values of q and p within the boundaries of the inequality we can take a linear combination of these distributions with weights $(3p - q)/2$ and $1 - (3p - q)/2$, chosen such that the weighed probabilities add to one, and obtain the joint probability distribution:

$$\begin{aligned} x &= \left(1 - \frac{3p - q}{2}\right) \frac{1 - q}{3}, \\ y &= \left(\frac{3p - q}{2}\right) \frac{q}{3}, \\ z &= \left(1 - \frac{3p - q}{2}\right) q, \\ w &= \frac{3p - q}{2} (1 - q), \end{aligned}$$

which proves that if the inequalities are satisfied a joint probability distribution exists, and therefore a noncontextual hidden variable as well, thus completing the proof. The generalization to the asymmetric case is tedious but straightforward.

As a consequence of the inequalities above, one can show that the correlations present in the GHZ state are so strong that even if we allow for experimental errors, the non-existence of a joint distribution can still be verified [Barros and Suppes 2000].

Corollary Let **A**, **B**, and **C** be three ± 1 random variables such that
(i) $E(\mathbf{A}) = E(\mathbf{B}) = E(\mathbf{C}) \geq 1 - \epsilon$,
(ii) $E(\mathbf{ABC}) \leq -1 + \epsilon$,
where ϵ represents a decrease of the observed GHZ correlations due to experimental errors. Then, there cannot exist a joint probability distribution of **A**, **B**, and **C** if

$$\epsilon < \frac{1}{2}. \tag{16}$$

Proof: To see this, let us compute the value of F define above. We obtain at once that

$$F = 3(1 - \epsilon) - (-1 + \epsilon).$$

But the observed correlations are only compatible with a noncontextual hidden variable theory if $F \leq 2$, hence $\epsilon < \frac{1}{2}$. Then, there cannot exist a joint probability distribution of **A**, **B**, and **C** satisfying (i) and (ii) if

$$\epsilon < \frac{1}{2}. \tag{17}$$

From the inequality obtained above, it is clear that any experiment that obtains GHZ-type correlations stronger than 0.5 cannot have a joint probability distribution. For example, the recent experiment made at Innsbruck [Bouwemeester et al. 1999] with three-photon entangled states supports the quantum mechanical result that no noncontextual hidden variable exists that explain their correlations [Barros and Suppes 2000]. Thus, with this reformulation of the GHZ theorem it is possible to use strong, yet imperfect, experimental correlations to prove that a noncontextual hidden-variable theory is incompatible with the experimental results.

3 Upper and Lower Probabilities and the GHZ theorem

We saw at the previous section that quantum mechanics does not allow, for some cases, the definition of a joint probability distribution for all the observables. However, if we weaken the probability axioms, it is possible to prove that one can find a consistent set of upper probabilities for the events [Suppes and Zanotti 1991]. Upper probabilities are defined in the following way. Let Ω be a nonempty set, F a boolean algebra on Ω and P^* a real valued function on F. Then the triple (Ω, F, P^*) is an *upper probability* if for all ξ_1 and ξ_2 in F we have that

(i) $0 \leq P^*(\xi_1) \leq 1$,

(ii) $P^*(\emptyset) = 0$,

(iii) $P^*(\Omega) = 1$,

and if ξ_1 and ξ_2 are disjoint, i.e. $\xi_1 \cap \xi_2 = \emptyset$, then

(iv) $P^*(\xi_1 \cup \xi_2) \leq P^*(\xi_1) + P^*(\xi_2)$.

As we can see, this last property weakens the standard axioms for probability, as one of the consequences of these axioms is that it may be true, for an upper probability, that

$$\xi_1 \subseteq \xi_2 \text{ and } P^*(\xi_1) > P^*(\xi_2),$$

a quite nonstandard property. In a similar way, *lower probabilities* are defined as satisfying the triple (Ω, F, P_*) such that for all ξ_1 and ξ_2 in F we have that

(i) $0 \leq P_*(\xi_1) \leq 1$,

(ii) $P_*(\emptyset) = 0$,

(iii) $P_*(\Omega) = 1$,

and if ξ_1 and ξ_2 are disjoint, i.e. $\xi_1 \cap \xi_2 = \emptyset$, then

(iv) $P_*(\xi_1 \cup \xi_2) \geq P_*(\xi_1) + P_*(\xi_2)$.

Let us see how upper and lower probabilities can be used to obtain joint upper and lower probability distributions. We can start with the standard Bell's variables $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, where each random variable represents a different angles for the Stern-Gerlach apparatus (we follow the example in [Suppes and Zanotti 1991]). In the experimental setup used by Bell, a two-particle system with entangled spin state was used, and for that reason we can only measure two variables at the same time. However, since they are spin measurements, we have the constraint

$$P(\mathbf{X}=1) = P(\mathbf{Y}=1) = P(\mathbf{Z}=1) = \frac{1}{2}.$$

The question that Bell posed is whether we can fill the missing values of the data table in a way that is consistent with the correlations given by quantum mechanics for the pairs of variables, that is, $E(\mathbf{XY})$, $E(\mathbf{XZ})$, $E(\mathbf{YZ})$. It is well known that for some sets of angles, the joint probability distribution of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ exists, while for other set of angles it does not exist. We can prove that the joint doesn't exist in the following way. We start with the values for the correlations used by Bell:

$$E(\mathbf{XY}) = -\frac{\sqrt{3}}{2}, \tag{18}$$

$$E(\mathbf{XZ}) = -\frac{\sqrt{3}}{2}, \tag{19}$$

$$E(\mathbf{YZ}) = -\frac{1}{2}. \tag{20}$$

The correlations above correspond to the angles $\widehat{\mathbf{XY}} = 30^{\mathrm{O}}$, $\widehat{\mathbf{YZ}} = 30^{\mathrm{O}}$ and $\widehat{\mathbf{XZ}} = 60^{\mathrm{O}}$ for the detectors, and require that

$$\begin{aligned} E(\mathbf{XY}) &= E(\mathbf{XY}|\mathbf{Z}=1)P(\mathbf{Z}=1) + E(\mathbf{XY}|\mathbf{Z}=-1)P(\mathbf{Z}=-1), \\ E(\mathbf{XZ}) &= E(\mathbf{XZ}|\mathbf{Y}=1)P(\mathbf{Y}=1) + E(\mathbf{XZ}|\mathbf{Y}=-1)P(\mathbf{Y}=-1), \\ E(\mathbf{YZ}) &= E(\mathbf{YZ}|\mathbf{X}=1)P(\mathbf{X}=1) + E(\mathbf{YZ}|\mathbf{X}=-1)P(\mathbf{X}=-1), \end{aligned}$$

which can be written as

$$2E(\mathbf{XY}) = E(\mathbf{XY}|\mathbf{Z}=1) + E(\mathbf{XY}|\mathbf{Z}=-1), \tag{21}$$

$$2E(\mathbf{XZ}) = E(\mathbf{XZ}|\mathbf{Y}=1) + E(\mathbf{XZ}|\mathbf{Y}=-1), \tag{22}$$

$$2E(\mathbf{YZ}) = E(\mathbf{YZ}|\mathbf{X}=1) + E(\mathbf{YZ}|\mathbf{X}=-1), \tag{23}$$

because $P(\mathbf{Z}=1) = P(\mathbf{Z}=-1)$, etc. Symmetry requires that

$$E(\mathbf{XY}|\mathbf{Z}=1) = E(\mathbf{YZ}|\mathbf{X}=1), \tag{24}$$

$$E(\mathbf{XY}|\mathbf{Z}=-1) = E(\mathbf{YZ}|\mathbf{X}=-1) \tag{25}$$

and if we use the requirement that all probabilities must sum to one we have six equations and six unknown conditional expectations. It is easy to see that

the system of linear equations (21)—(25) does not have a solution for the correlations shown in (18), hence no joint probability distribution exists. What happened? The correlations are too strong for us to fill up a table with all the experimental results, including the ones that did not occur. One extreme example can be obtained if we use the extreme case of correlation one expectations, given by

$$\begin{aligned} E(\mathbf{XY}) &= -1, \\ E(\mathbf{YZ}) &= -1, \\ E(\mathbf{XZ}) &= -1, \end{aligned}$$

where once again no joint probability distribution exists.

What changes with upper probabilities? The system of linear equations (21) becomes a system of inequalities:

$$2E^*(\mathbf{XY}) \geq E^*(\mathbf{XY}|\mathbf{Z}=1)+E^*(\mathbf{XY}|\mathbf{Z}=-1), \tag{26}$$

$$2E^*(\mathbf{XZ}) \geq E^*(\mathbf{XZ}|\mathbf{Y}=1)+E^*(\mathbf{XZ}|\mathbf{Y}=-1), \tag{27}$$

$$2E^*(\mathbf{YZ}) \geq E^*(\mathbf{YZ}|\mathbf{X}=1)+E^*(\mathbf{YZ}|\mathbf{X}=-1), \tag{28}$$

plus the symmetry

$$E^*(\mathbf{XY}|\mathbf{Z}=1) = E^*(\mathbf{YZ}|\mathbf{X}=1), \tag{29}$$

$$E^*(\mathbf{XY}|\mathbf{Z}=-1) = E^*(\mathbf{YZ}|\mathbf{X}=-1), \tag{30}$$

and the fact that the sum of all upper probabilities must be greater or equal than one. It is straightforward to obtain solutions to (26)–(30), and then we can find upper probabilities that are consistent with the conditional expectations.

The following theorem shows that the GHZ theorem fail if we allow lower probabilities.

Theorem 3 Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be three ± 1 random variables and let
(i) $E_*(\mathbf{A}) = E(\mathbf{A}) = 1$,
(ii) $E_*(\mathbf{B}) = E(\mathbf{B}) = 1$,
(iii) $E_*(\mathbf{C}) = E(\mathbf{C}) = 1$,
(iv) $E_*(\mathbf{ABC}) = E(\mathbf{ABC}) = -1$.
Then, there exist a lower joint probability distribution that is compatible with (i)—(iv).

Proof: We will prove this theorem by explicitly constructing a lower joint probability distribution. First, we note that

$$E_*(\mathbf{A}) = P_*(a) - P_*(\overline{a}) = 1,$$

$$E_*(\mathbf{B}) = P_*(b) - P_*(\overline{b}) = 1,$$

$$E_*(\mathbf{C}) = P_*(c) - P_*(\overline{c}) = 1,$$

and hence

$$P_*(a) = 1, \qquad P_*(\overline{a}) = 0, \tag{31}$$
$$P_*(b) = 1, \qquad P_*(\overline{b}) = 0, \tag{32}$$
$$P_*(c) = 1 \qquad P_*(\overline{c}) = 0. \tag{33}$$

From the definition of lowers and from (31)–(33) we have

$$P_*(abc) + P_*(a\overline{b}c) + P_*(ab\overline{c}) + P_*(a\overline{b}\overline{c}) \quad \leq \quad 1, \tag{34}$$
$$P_*(abc) + P_*(\overline{a}bc) + P_*(ab\overline{c}) + P_*(\overline{a}b\overline{c}) \quad \leq \quad 1, \tag{35}$$
$$P_*(abc) + P_*(\overline{a}bc) + P_*(a\overline{b}c) + P_*(\overline{a}\overline{b}c) \quad \leq \quad 1, \tag{36}$$

and from (iv)

$$P_*(abc) + P_*(\overline{a}\overline{b}c) + P_*(a\overline{b}\overline{c}) + P_*(\overline{a}b\overline{c}) + \tag{37}$$
$$-P_*(\overline{a}bc) - P_*(a\overline{b}c) - P_*(ab\overline{c}) - P_*(\overline{a}\overline{b}\overline{c}) \quad = \quad -1. \tag{38}$$

The lowers must also be superadditive in the whole probability space, and we have

$$P_*(abc) + P_*(\overline{a}\overline{b}c) + P_*(a\overline{b}\overline{c}) + P_*(\overline{a}b\overline{c}) + \tag{39}$$
$$P_*(\overline{a}bc) + P_*(a\overline{b}c) + P_*(ab\overline{c}) + P_*(\overline{a}\overline{b}\overline{c}) \quad \leq \quad 1. \tag{40}$$

From (38) and (40) we have

$$P_*(abc) = P_*(\overline{a}\overline{b}c) = P_*(a\overline{b}\overline{c}) = P_*(\overline{a}b\overline{c}) = 0$$

and the system reduces to

$$P_*(a\overline{b}c) + P_*(ab\overline{c}) \quad \leq \quad 1, \tag{41}$$
$$P_*(\overline{a}bc) + P_*(ab\overline{c}) \quad \leq \quad 1, \tag{42}$$
$$P_*(\overline{a}bc) + P_*(a\overline{b}c) \quad \leq \quad 1, \tag{43}$$
$$P_*(\overline{a}bc) + P_*(a\overline{b}c) + P_*(ab\overline{c}) + P_*(\overline{a}\overline{b}\overline{c}) \quad = \quad 1. \tag{44}$$

A possible solution for the system (41)–(44) is

$$P_*(\overline{a}bc) = P_*(a\overline{b}c) = P_*(ab\overline{c}) \quad = \quad \frac{1}{3}$$
$$P_*(\overline{a}\overline{b}\overline{c}) \quad = \quad 0,$$

as we wanted to prove. In a similar way, we have the following:

Theorem 4 Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be three ± 1 random variables and let
(i) $E^*(\mathbf{A}) = E(\mathbf{A}) = 1$,
(ii) $E^*(\mathbf{B}) = E(\mathbf{B}) = 1$,
(iii) $E^*(\mathbf{C}) = E(\mathbf{C}) = 1$,
(iv) $E^*(\mathbf{ABC}) = E(\mathbf{ABC}) = -1$.
Then, there exist an upper probability distribution that is compatible with (i)—(iv).

Proof: Similar to the proof for the lower.

We note that the nonmonotonic upper and lower probabilities shown to exist in Theorems 3 and 4 do not, because of their nonmonotonicity, satisfy the usual definitional relation between upper and lower probabilities, for any event A:

$$P^*(A) = 1 - P_*(\overline{A}).$$

4 Final Remarks

To apply the upper probabilities to the GHZ theorem, we gave a probabilistic random variable version of it. We then showed that, if we use upper probabilities, the GHZ theorem does not hold anymore, and hence the inconsistencies cannot be proved to exist for the upper probabilities. Such upper probabilities are a natural way to deal with contextual problems in statistics. Whether they lead to fruitful theoretical developments in a new direction is, however, an open question.

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